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Type II string theory and modularity*

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Abstract: This paper, in a sense, completes a series of three papers. In the previous two [1, 2], we have explored the possibility of refining the K-theory partition function in type II string theories using elliptic cohomology. In the present paper, we make that more concrete by defining a fully quantized free field theory based on elliptic cohomology of 10-dimensional spacetime. Moreover, we describe a concrete scenario how this is related to compactification of F-theory on an elliptic curve leading to IIA and IIB theories. We propose an interpretation of the elliptic curve in the context of elliptic cohomology. We discuss the possibility of orbifolding of the elliptic curves and derive certain properties of F-theory. We propose a link of this to type IIB modularity, the structure of the topological lagrangian of M-theory, and Witten's index of loop space Dirac operators. The discussion suggests a S^1-lift of type IIB and an F-theoretic model for type I obtained by orbifolding that for type IIB.

Keywords: Superstrings and Heterotic Strings, F-Theory, String Duality, M-Theory.

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Contents

1. Introduction

The main purpose of this paper is to complete, at least in some sense, our investigation [1, 2] of a refinement of the K-theory RR partition function of type II string theory [3, 4]. The RR K-theory partition function is obtained from the observation that $K^0(X)$ and $K^1(X)$ classifies the RR sources in type IIA and IIB string theory on a 10-manifold X [5, 6]. The K-theory partition function is a theta function obtained from quantizing essentially the free field theory on those sources, i.e. where the lagrangian is essentially the hermitean metric on a field strength which is set up in such a way that the phase term of the lagrangian can be thought of as an index (see [4]). The main result of [4] is that in type IIA string theory, this partition function agrees with the partition function of M-theory compactified on S^1 where the effective action is taken to be the Chern-Simons term together with correct normalization and a 1-loop correction term, which makes the phase again an index. This time, however, the index is a combination of an E_8-index and a Rarita-Schwinger index on a 12-manifold Z^{12} which cobords $X \times S^1$. An extension of part of the construction to twisted K-theory, i.e. to include NSNS background, was studied in [7].

In [1], we observed that an anomaly $W_7(X)$ detected in [4] on both the IIA and M-theory sides coincides with the anomaly of orientation with respect to elliptic cohomology. This led us to propose in [1] a refinement of the K-theory partition function, which would be based on elliptic cohomology. When fully quantizing the theory associated with that partition function, we encountered a refinement of the obstruction $W_7(X)$ to $w_4(X)$ (one has $W_7(X) = \beta Sq^2(w_4(X))$). This suggests that the elliptic partition function must be related to a scenario where type II string theory is unified with type I and heterotic, where a 4-dimensional obstruction is detected — such obstruction is not known in type II theory per se.

Elliptic cohomology is a certain refinement of K-theory which is introduced in topology, and which has the striking property that its coefficients (homotopy groups, or cohomology groups of a point) consist of modular forms, of weight $k/2$ where k is dimension. This led us to propose in [2], after eliminating some simpler scenarios, that the elliptic cohomology partition function may be a step toward solving the puzzle of [4] related to IIB modularity: when one writes the K-theory partition function for type IIB, it does not seem to accomodate a modularity in the presence of an H_3 source. It is remarked in [4] and further investigated in [2] that twisted K-theory, which is the first approach which comes to mind, does not solve the problem.

The papers [1, 2] left unanswered the question where the elliptic cohomology source partition function of type II string theory really comes from. In [4], the IIA partition function is linked to M-theory compactified on a circle. What, if any, is the analogous link for the elliptic cohomology partition function? In this paper, we attempt to answer that question, and derive some implications from the answer.

The scenario we propose is that the elliptic cohomology partition function is related to compactification of F-theory on an elliptic curve E, which is a theory first suggested by Vafa [8]. We propose that the modularity of elliptic cohomology, which makes the partition function itself modular, comes from modularity in $H^1(E)$, i.e. from the moduli parameter of the elliptic curve E. This is in fact somewhat linked to the purely mathematical paper [9], in which it was derived that in developing a purely mathematical link between elliptic cohomology and conformal field theory, modularity of elliptic cohomology is related to an elliptic curve in spacetime, not merely to a genus one worldsheet.

In the present paper, the link in fact goes one step further: in the same way in which the action of the free approximation to the RR sector of type II theory is related to the $\hat{A}$-genus, the elliptic cohomology form of the RR sector of the theory is related to the Witten genus. Why is that? We propose an explanation. In [10], Witten shows that the Witten genus is related to index of elliptic operators on loop space. Using a standard lifting [11] of the M-theory action to F-theory [8] (cf. [12] for the reduction), we propose that in F-theory compactified on a circle, the phase factor analogous to that analyzed in [4] may be obtained as loop versions of the E_8 and Rarita-Schwinger indices. We view the loop group bundles [13, 14, 7] as coming from bundles over the loop space of spacetime. Thus, we possibly link modularity in four different places: S-duality in type IIB string theory, elliptic cohomology (spacetime aspect), the Witten genus and F-theory fiber.

There is one caveat to our story: the modularity of elliptic cohomology is not entirely anomaly-free, and accordingly the modular forms are really just automorphic forms of a certain level ($= 3$ if we focus on integrality at the prime 2). There is a theory constructed by Mike Hopkins and known as TMF (topological modular forms, the connective form of the theory is known as tmf) which remedies the difficulty: this theory has a complete anomaly-free modularity. However, the price for that is that it is again a much more complicated generalized cohomology theory, which can no longer be called an elliptic cohomology theory — it is obtain from elliptic cohomology theory by a procedure which we could compare to the physical procedure of orbifolding.

Indeed, this orbifolding seems to correspond to orbifolding in physics literally. Orbifolding type IIB in the worldsheet sense (reversing chiralities) leads to type I string theory. Orbifolding S^1-compactified M-theory with respect to the 11-th dimension in spacetime leads to Hořava-Witten M-theory [15]. We deduce from this a relation between this type of worldsheet and spacetime orbifolding. In F-theory, we further predict a more complicated orbifolding with respect to (roughly) the group $\mathrm{SL}(2, \mathbb{Z}/2)$, which should produce an "ideal F-theory" governed by TMF, which would have a complete, anomaly free modularity. We do not however work out this optimal scenario in detail, and in most of this paper still use just ordinary elliptic cohomology instead.

The Chern-Simons part of the action of M-theory was used by Witten and rewritten as in a symmetric way as a cubic expression in the four-form [16]. One wonders whether there is a general mathematical reason for such a structure, beyond just being able to use some form of Stokes' theorem. We show that the lifted Chern-Simons term can be written as a Massey triple product and the one-loop term can be explained as being a part of the Massey product indeterminacy.

The lagrangian of the ultimate twelve dimensional theory is not completely worked out in the present paper. There are at least two sources of topological terms in such lagrangian, one of them which should be related to M-theory upon compactification on S^1, another which should be related to M-theory by cobordism. However, it is possible that a Massey product device similar to the one mentioned above can also be used to unify these situations — we make a comment to that effect.

The present paper is organized as follows: in section 2, we revie basic features of IIB modularity from a classical and quantum-mechanical point of view. In section 3, we review the lagrangians of known theories in 12 dimensions. In section 4, we present our evidence for topological modular forms from the IIB modularity question and also from the cobordism approach to M-theory [16]. In section 5, we give our main explanations about the elliptic cohomology partition function and its relation to F-theory and the Witten genus. Finally, in section 6, we give some general comments on what would be needed to discuss F-theory at physical signatures. In order to make the paper more self-contained and more accessible to physicists, we included a brief appendix on topological modular forms.

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2. Type IIB and modularity

In this section we review the basic features of type IIB supergravity and string theory that will be relevant for our later discussions.

The bosonic field content of type IIB supergravity is: metric g, two scalars ϕ and χ, a complex 3-form field strength G_3 and a real self-dual five-form field strength F_5. The fermionic content is: two gravitini ψ^i ($i = 1, 2$) of the same chirality, i.e. sections of $S(X)^{\pm} \otimes (TX - 2\mathcal{O})$ (with the same choice of sign), and two dilatini of the opposite chirality to the gravitini, i.e. $\lambda^i \in \Gamma[S(X)^{\mp}]$. The two scalars parametrize an upper half plane $\mathcal{H} = \mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$. In a fixed U(1) gauge, the global $\mathrm{SL}(2,\mathbb{R})$ induces on the fields, collectively Φ, a U(1) transformation that depends on their U(1) charge q_Φ, as [17]

$$\Phi \longrightarrow \Phi \left(\frac{c\overline{\tau} + d}{c\tau + d}\right)^{\frac{q_\Phi}{2}} . \tag{2.1}$$

The $\mathrm{SL}(2,\mathbb{R})$ symmetry is broken down to the local discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ by nonperturbative quantum effects. The arithmetic subgroup is conjectured to be an exact symmetry of type IIB string theory. Its action factorizes into a projective action on the complex scalar τ and a charge-conjugation that reverses the signs of the two 2-forms and leaves τ invariant.

Strings with fractional charge do not exist and so the type IIB string must be $\mathrm{SL}(2,\mathbb{Z})$ invariant [18, 19]. The action can be written in a manifestly SL(2)-invariant way as

$$\begin{aligned} S_{IIB} = {}& \frac{1}{2\kappa_{10}^2}\int_{X^{10}} d^{10}x\sqrt{-g}\left[\mathcal{R} - \frac{1}{4}\,\mathrm{Tr}(\partial\mathcal{M}\mathcal{M}^{-1})^2 - \frac{1}{12}\bar{H}^T_{\mu\nu\rho}\mathcal{M}\bar{H}^{\mu\nu\rho} - \frac{1}{4}\tilde{F}_5^2\right] - \\ & - \frac{1}{8\kappa_{10}^2}\int_{X^{10}} C_4 \wedge H^i \wedge H^j \epsilon_{ij}\,, \end{aligned} \tag{2.2}$$

where $\mathcal{M}$ is the metric on the coset $\mathrm{SL}(2)/\,\mathrm{U}(1)$ (i.e. the upper half plane) given by

$$\mathcal{M} = \frac{1}{\mathrm{Im}(\tau)}\begin{pmatrix} |\tau|^2 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & 1 \end{pmatrix}, \tag{2.3}$$

$\bar{H} = \binom{H^1}{H^2}$ is the doublet of three-forms with $H^i = (H_3, F_3)$. The five-form $\tilde{F}_5$ is an RR field strength modified by the RR and NS three-forms, i.e.

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{2.4}$$

with $F_5 = dC_4$, which can be written as

$$\tilde{F}_5 = F_5 + \frac{1}{2}\epsilon_{ij}B^i \wedge H^j\,, \qquad i, j = \{1, 2\}\,, \tag{2.5}$$

and C_4 is the SL(2)-invariant RR 4-form potential. This way we see that the two scalars, namely the dilaton from the NS sector and the axion from the RR sector, can be viewed as the coordinates on the upper half-plane. So the modular parameter is built out of the dilaton ϕ and the axion (= 0-form RR potential), $\tau = C_0 + ie^{-\phi}$.

The above action is invariant under the $\mathrm{SL}(2,\mathbb{Z})$ transformations

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T \tag{2.6}$$

$$\overline{H}' = (\Lambda^T)^{-1}\overline{H} \tag{2.7}$$

with the metric in the Einstein frame being invariant, $g'_{\mu\nu} = g_{\mu\nu}$, where the group element is $\Lambda = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $ad - bc = 1$.

Alternatively, one can choose to use complex differential forms and write the effective action of type IIB string theory in the $\mathrm{SL}(2,\mathbb{Z})$-invariant form (in the Einstein frame, see e.g. [20])

$$\begin{aligned} S_{IIB} = &\frac{1}{2\kappa_{10}^2}\int_{X^{10}} d^{10}x\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}\frac{\partial_M\tau\partial^M\bar{\tau}}{(\mathrm{Im}\,\tau)^2} - \frac{1}{12}G_3\wedge *\overline{G}_3 - \frac{1}{4}\tilde{F}_5^2\right] + \\ &+\frac{1}{8i\kappa_{10}^2}\int_{X^{10}} C_4\wedge G_3\wedge\overline{G}_3 + S_{(2p+1)-\mathrm{branes}}\,. \end{aligned} \tag{2.8}$$

Here the RR field strength F_3 and the NS field H_3 are grouped as $\mathrm{SL}(2,\mathbb{Z})$ doublet into the invariant complex field

$$G_3 = \frac{1}{\sqrt{\mathrm{Im}\,\tau}}(F_3 - \tau H_3) \tag{2.9}$$

and similarly for the complex conjugate field,

$$\overline{G}_3 = \frac{1}{\sqrt{\mathrm{Im}\,\tau}}(F_3 - \overline{\tau} H_3)\,. \tag{2.10}$$

The self-duality for $\tilde{F}_5$ cannot be seen at the level of the above action but has to be imposed as an extra condition on the equations of motion. The action is obviously invariant under $\mathrm{SL}(2,\mathbb{Z})$ transformations.

The S-duality transformation is the subset of the above $\mathrm{SL}(2,\mathbb{Z})$ transformations given by $a = d = 0$ and $b = -c = 1$, so that the fields transform as

$$\begin{aligned} \tau &\to -1/\tau \\ B_2 &\to C_2 \\ C_2 &\to -B_2\,, \end{aligned} \tag{2.11}$$

and again the metric and the five form are left invariant.

The moduli space of scalar fields is then $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$. The supersymmetry algebra has an automorphism group, a continuous U(1) R-symmetry that rotates the supercharges, and this is broken down to a discrete subgroup [21] $\mathbb{Z}_4 = \mathrm{SL}(2,\mathbb{Z})\cap\mathrm{U}(1)$ that interchanges the two supercharges and reverses the spatial worldsheet direction.

The $\mathbb{Z}_4$ symmetry (see e.g. [22]) generated by the elements $a = 0$, $b = 1$, $c = -1$, $d = 0$ inverts the modular parameter τ as $\tau \to \frac{-1}{\tau}$, so that for vanishing axion $C_0 = 0$, this inverts the coupling constant $e^{-\phi} \to e^{\phi}$, which can be interpreted as the weak/strong coupling duality (S-duality). This $\mathbb{Z}_4$ symmetry also acts on the NS and RR 2-forms as $B_2^{(1)} \to -B_2^{(2)}$ and $B_2^{(2)} \to B_2^{(1)}$, so that G_3 and $\bar{G}_3$ are interchanged and $\tilde{F}_5$ is of course still invariant. This duality also acts on the metric in the string frame, and that is why one has to use the Einstein frame to get a duality-invariant action. Applying $\mathbb{Z}_4$ twice gives a $\mathbb{Z}_2$ with almost trivial effect, in the sense that it leaves τ invariant but changes the sign of the two 2-forms $B_2^{(i)}$.

There is no one-loop correction in type IIB in ten dimensions analogous to the term $\int B_2 \wedge X_8(R)$ in type IIA [23, 24]. The nonperturbative result for type IIB is [25, 17]

$$L = f(\tau, \bar{\tau}) \left(I_1 - \frac{1}{8} I_2 \right) \tag{2.12}$$

where $f(\tau, \bar{\tau})$ is a modular form in $\tau = C_0 + ie^{-\phi}$, and [26][1]

$$\begin{aligned} I_1 &= t_8 t_8 R^4 + \frac{1}{2} \epsilon_{10} t_8 B_2 R^4 \\ I_2 &= -\epsilon_{10} \epsilon_{10} R^4 + 4 \epsilon_{10} t_8 B_2 R^4 \end{aligned} \tag{2.13}$$

so that the term in L involving B_2 cancels out and one is left with the pure R^4 term. There is also a similar perturbative result at tree level and one-loop. For type IIA there is another term in L with a + sign between I_1 and I_2, which leads to nonzero $B_2 R^4$ term [27] (see [28] for details). This is still compatible [29] with type II T-duality, because of radius dependence of the corresponding term in nine dimensions.

The $\mathrm{SL}(2, \mathbb{Z})$ symmetry of type IIB string theory in nine dimensions can be interpreted as a geometric symmetry of M-theory compactified on a torus T^2 [30–32]. This way there are three scalar fields corresponding to the moduli of the torus (along directions 9 and 11) given by the volume $V = R_9 R_{11}$ and the complex structure $\omega = \omega_1 + i\omega_2 = C_1 + iR_9/R_{11}$, where the metric on the torus is

$$G_{IJ} = \frac{V}{\omega_2} \begin{pmatrix} |\omega|^2 & \omega_1 \\ \omega_1 & 1 \end{pmatrix}. \tag{2.14}$$

By T-duality $R_A \leftrightarrow 1/R_B$, ω is identified with τ of type IIB theory, and thus manifests itself as the S-duality in type IIB $e^{-\phi} \leftrightarrow e^{\phi}$.

All R^4 one-loop terms can be obtained from one-loop terms in M-theory [33]. Such terms contain factors that are of the form

$$\begin{aligned} A_{R^4} &\sim \int dt \sum_{l_1, l_2} \exp(-t G_{IJ} l_I l_J) \\ &\sim \int dt \sum_{l_1, l_2} \exp\left(-\frac{t}{V} \frac{|m + n\omega|^2}{\omega_2} \right). \end{aligned} \tag{2.15}$$

A double Poisson resummation converts the sum over the Kaluza-Klein charges (m, n) to a sum over the winding modes $(\hat{m}, \hat{n})$ of a worldline along the two cycles of T^2, and the

[1] Here t_8 is the usual rank eight tensor that shows up in higher order corrections, ϵ_{10} is the antisymmetric constant tensor, and R^4 is a certain quartic polynomial in the curvature tensor.

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gaussian integral gives terms proportional to the nonholomorphic modular form of weight zero [25, 17, 34, 35] (see [33] for an overview)

$$\sum_{(\hat{m},\hat{n})\neq(0,0)} \frac{{\omega_2}^{3/2}}{|\hat{m}+\hat{n}\omega|^3}\,. \tag{2.16}$$

3. Theories in twelve dimensions

Let us now begin looking at what sectors of F-theory are actually known. Quite a lot is already in the literature. One can get hints from eleven-dimensional M-theory and ten-dimensional type IIB string theory that there is a theory (or theories)[2] in twelve dimensions that is (are) playing a role in the topology and the dynamics of those theories. One can think of two such theories, the manifolds on which they are defined we take to be Z^{12} and V^{12}, respectively.

First, there is the twelve-dimensional coboundary theory that Witten introduced [16] to rewrite the Chern-Simons term of M-theory in terms of the index of the E_8-coupled Dirac operator and the index of the Rarita-Schwinger operator. The topological part of the low energy limit of M-theory, namely eleven-dimensional supergravity, is captured by the Chern-Simons term and the one-loop gravitational correction term,

$$\frac{1}{6}\int_{Y^{11}} C_3\wedge G_4\wedge G_4 - C_3\wedge I_8 \tag{3.1}$$

where I_8 is a polynomial in the curvature of Y^{11} whose class is given in terms of the Pontrjagin class and the string class as $[I_8]=\frac{p_2-\lambda^2}{48}$. The lift of this action to the twelve-dimensional manifold Z^{12} (where $Y^{11}=\partial Z^{12}$) is given by [16]

$$\frac{1}{6}\int_{Z^{12}} G_4\wedge G_4\wedge G_4 - G_4\wedge I_8 \tag{3.2}$$

by directly using Stokes' theorem.[3] A priori this theory has no connection to type IIB. However, we will show later that there is in fact such a connection.

Second, there is the "standard" F-theory [8], which is the lift of type IIB via an elliptic curve. The complex structure of the elliptic curve is varying over the type IIB base. In contrast to conventional type IIB compactifications where τ, as a physical parameter, is taken to be constant. One can relate type IIB on a manifold X^{10} to F-theory on an elliptically fibered manifold with base X^{10}. A choice of section is usually required [36] for the elliptically fibered manifold, i.e. a choice of an embedded base manifold.

To be compatible with dualities, this theory can also be considered as the lift of M-theory via a circle. If we choose to start from M-theory, then we lift the action (3.1) via a circle S^1 to[4]

$$\frac{1}{6}\int_{V^{12}} A_4\wedge G_4\wedge G_4 - A_4\wedge I_8\,, \tag{3.3}$$

[2]In this paragraph we use the term "theory" rather loosely and we do not yet specify the dynamics (nor claim a full construction of course).

[3]Of course this is not as trivial as it seems because it requires the vanishing of the relevant spin cobordism groups. Happily, this satisfied.

[4]Such terms were proposed in [11] in the context of Calabi-Yau compactifications.

where A_4 is a 4-form potential which is the lift to twelve dimensions of the 3-form potential C_3 of M-theory in eleven dimensions. On can view C_3 in turn as the contraction of one index of A_4, i.e. $C_3 = i_* A_4$. This lagrangian has been essentially considered in [12], and global manipulations of this type for circle bundles have been considered in [7].

Looking for field theory in F-theory as a circle bundle on M-theory leads to certain puzzles and we cannot claim that all sectors of F-theory arise in this way (accordingly, the F-theory lagrangian may need other terms which we do not yet know). To see this, for example IIA string theory should be a compactification of M-theory and IIB string theory should be a compactification of F-theory, but as far as known so far, IIA and IIB spacetimes can have different homotopy types. This seems contradicted by proposing a simple relation between M-theory and F-theory via S^1-compactification.

While we do not have a definitive answer to this problem, there are two ways we can deal with it in the present paper: first, our main interest is a free field theory based on elliptic cohomology, which approximates a certain refinement of the partition function in type II string theory. For this elliptic field theory to exist, a stronger condition ($w_4 = 0$) is required than the known conditions for consistence of type II theories. The condition we use is the same for type IIA and IIB, so it can be argued that duality is not violated in our setting (it is at present unknown if the stronger condition is simply an artefact of our model, or if it expresses some intrinsic new restriction on type II strings).

The second possible approach is to deal with IIA and IIB separately. We shall discuss this in more detail below, and in fact shall see evidence that different physical signatures may arise in both cases. In this approach, the lagrangian (3.3) is valid for the sector of F-theory which contains M-theory and type IIA, and the precise lagrangian for IIB remains to be determined (however, should be related, since, as we shall see, the present lagrangian can be interpreted in a way as to contain IIB fields). It should be also mentioned that [12] consider certain projections to reconcile on-shell states between F-theory and M-theory/type IIB. While we believe this might be possible in our formulation (see end of section 5.3), we do not attempt a construction, as it seems out of the scope of the present paper.

4. Evidence for TMF

4.1 Type IIB and TMF

Let us now consider again the equation (2.9). From what we learned in previous investigations, it is likely correct to say that the field strength G_3 should live in a generalized cohomology theory. For example, when analyzing the IIB partition function, Witten found that $F_3 \in K^1(X)$. K-theory, on its own, of course does not tell a modularity story, and one needs to solve the puzzle of what happens in the presence of H_3. Some aspects of this were considered in [2]. But from the point of view of (2.9), it seems that if we want G_3 to live in a generalized cohomology theory, then τH_3, F_3 must coexist in the same theory. We conjectured in [2] that this theory should be the theory of topological modular forms, tmf, the coefficients of which are, at least rationally, holomorphic (chiral) modular forms — see appendix for a brief review. However, even then, what should one do about the factor $1/\sqrt{\operatorname{Im}\tau}$? This scaling factor is troublesome from the point of view of algebraic topology, since it is not chiral and therefore does not occur among the kinds of modular forms described by tmf.

If we are to lift our fluxes to tmf, we must proceed one chirality at a time, and therefore see no choice but to drop the $1/\sqrt{\operatorname{Im}\tau}$ factor. Thus, we consider

$$\tilde{G}_3 = F_3 - \tau H_3\,. \tag{4.1}$$

This, of course, now is a flux with modular weight -1, i.e. we have if we denote by $\tilde{G}_3'$ the expression obtained by replacing τ by

$$\tau' = \frac{(a\tau + b)}{(c\tau + d)}\,, \tag{4.2}$$

$$\tilde{G}_3' = \tilde{G}_3 \cdot (c\tau + d)^{-1}\,. \tag{4.3}$$

Now this has a striking implication to the dimension of this class, if it is to be lifted to tmf: in that theory, a class of modular weight k appears in $tmf^{2k}(X^{10})$. Therefore, our assumptions lead to

$$\tilde{G}_3 \in tmf^{-2}X^{10}. \tag{4.4}$$

This points to the 12-dimensional picture: suppose, in the simplest possible scenario following Vafa [8] that

$$V^{12} = X^{10} \times E \tag{4.5}$$

where E is an elliptic curve. Then let

$$\mu \in tmf^2(E) \tag{4.6}$$

be the generator (given by orientation). This then suggests introducting μ, instead of $1/\sqrt{\operatorname{Im}\tau}$, as the correct scaling factor of G_3, and passing to 12 dimensions: we have

$$\tilde{G}_3 \times \mu \in tmf^0\left(V^{12}\right)\,. \tag{4.7}$$

It is a surprise that the class ends up in dimension 0 and no odd number shows up here. However, note that the fiber E contains odd-degree non-torsion cohomological classes, so all kinds of shifts between even and odd are possible here. Modular classes of weight 0, however, must be in dimension 0.

We used here the statement that for a space X, classes in $tmf^k(X)$ are modular of weight $k/2$, which means that upon the transformation (4.2), the class transforms by introducing the factor $(c\tau + d)^{k/2}$. It is fair to point out that to make this rigorous mathematically, some discussion is needed. In fact, we will find it necessary to generalize to an elliptic cohomology theory E which is in general modular only with respect to some subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ (see below). So, we give the discussion in this context. The tmf discussion is analogous. The first question we must ask is what is τ mathematically? The answer is that τ appears only when we apply the forgetful map[5]

$$E^k(X) \to K^k(X)\left[\left[q^{1/24}\right]\right]\left[q^{-1/24}\right]\,. \tag{4.8}$$

[5]This is formal expansion of K-theory in the the power $q^{1/24}$ of the formal parameter q. The second set of paranthesis indicates that the generator $q^{1/24}$ is inverted. Such expansions relating elliptic cohomology to K-theory were used in [1] to interpret the elliptic refinement of the type IIA partition function.

Then one takes $q = exp(2\pi i\tau)$. The right hand side denotes power series in K-groups, with the parameter $q^{1/24}$ inverted. This map was discussed in our previous papers [1, 2]. On this level of coefficients, it is given simply by the fact that a modular form may be expanded in the modular parameter τ. For forms which are modular only with respect to a subgroup of the modular group, fractional powers of q are needed: in the case of complex-oriented cohomology, one encounters $q^{1/24}$.

But in addition to this, (4.8) must be suitably normalized. As explained in [1], one has a canonical map of generalized cohomology theories

$$E \to K\left[\left[q^{1/24}\right]\right]\left[q^{-1/24}\right] \tag{4.9}$$

whose induced map on coefficients (homotopy groups) makes the k-th homotopy group modular of weight $k/2$. This is not the correct normalization to use in (4.8), because then $\tilde{E}^0(S^k) = E^{-k}(*)$ would have modular weight $-k/2$, whereas we would like 0. To this end, we need to compose with some map which would multiply by some normalizing factor of weight $k/2$ in the k-th homotopy group. Such operation indeed exists, and it is the Adams operation[6]

$$\psi^\eta : K\left[\left[q^{1/24}\right]\right]\left[q^{-1/24}\right] \to K\left[\left[q^{1/24}\right]\right]\left[q^{-1/24}\right]. \tag{4.10}$$

Here η is the Dedekind function ($\Delta^{1/24}$ where Δ is the discriminant form), which, note, is a unit in $K[[q^{1/24}]][q^{-1/24}]$. Now we see that composing (4.9) with (4.10) gives the correct normalization of (4.8) for $k = 0$. For general k, if we simply delooped this map, we would be be in weight 0 instead of $k/2$, so we need to multiply the delooped map by η^k to get the correct normalization.

To summarize the results of this section, our conclusion confirms that if we want to seriously consider the modularity of the flux G_3 in tmf, the correct way is to introduce the normalization (4.7), and work in F-theory. We will see in the later sections that the picture described above may be overambitious: we do not know of a sector of F-theory which would really use tmf this way, and which would explain modularity of IIB with respect to the whole group. Nevertheless, we will see that the naive discussion given in this section is roughly correct.

4.2 Twelve dimensions and TMF

Let us dedicate one section to speculation about an F-theory which would be governed by the ideally modular elliptic cohomology theory tmf. As already remarked, we will see later that we will fall somewhat short of this goal, and will have to revert to less ideal elliptic cohomology theories and elliptic curves. Perhaps the "ideal theory" could be reached by some type of advanced orbifolding of the fiber E in (4.5), just as tmf in mathematics is constructed that way from elliptic cohomology. For now, however, let us make a few first observations about the field content of such ideal F-theory.

[6] In fact, a more precise discussion uses Ando power operations in elliptic cohomology, but we will not need that here.

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At least when tensored with $\mathbb{Q}$ (or more generally a field of characteristic 0), the coefficients (homotopy classes) of tmf are modular forms:

$$tmf_* \otimes \mathbb{Q} = \mathbb{Q}[g_2, g_3]\,. \tag{4.11}$$

Here recall that g_2, g_3 are the standard modular forms of weights 4 and 6, given by the Eisenstein series as

$$g_2(z) = \frac{4}{3}\pi^4 E_4(z) \tag{4.12}$$

$$g_3(z) = \frac{8}{27}\pi^6 E_6(z) \tag{4.13}$$

with the (normalized) Eisenstein series given by

$$E_k(z) = \frac{1}{2} \sum_{m,n\in\mathbb{Z};(m,n)=1} \frac{1}{(mz+n)^k} \tag{4.14}$$

where (m, n) denotes the greatest common divisor.

In particular, the notation has nothing to do with our previous notation for fluxes. Now as remarked above, in tmf, the dimension of a class is twice its modular weight, so rationally,

$$\begin{aligned} g_2 &\in tmf_8\,, \\ g_3 &\in tmf_{12}\,. \end{aligned} \tag{4.15}$$

Now the F-theory we are considering takes place on a tmf-orientable manifold Z^{12}, and the topological fluxes we consider are in its tmf-cohomology. Recall (cf. [2]) that the obstruction to tmf-orientability is

$$\lambda \in H^4(Z, \mathbb{Z}) \mod 24\,. \tag{4.16}$$

In any case, orientability implies that we have a class

$$u \in tmf^{12}(Z)\,, \tag{4.17}$$

so using (4.11), and the dimensions, we see that in non-negative dimensions, we have possible field strength sources u, ug_2, ug_3 in dimensions 12, 4, 0 (the dimension of the coefficients is subtracted from the dimension of a class in generalized cohomology). Note that this derivation is of course quite schematic, but on the other hand somewhat analogous to the derivation of the dimension of RR-sources in type II string theory from K-theory. Also, we have only considered tmf rationally. Delicate questions regarding the integrality of the proposed fields would have to be considered, specifically at the primes 2 and 3.

If we accept this, then we see there is a fundamental field strength in dimension 4. It is, of course, natural to conjecture that this is related to the field strength G_4 in the M-theory compactification of the appropriate sector of F-theory. We look at this next.

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4.3 Anomalies in type IIB and congruences

In principle there can be anomalies associated with the U(1) symmetry and with the SL$(2,\mathbb{Z})$. The U(1) anomaly [37] can be cancelled by adding the term [21]

$$S=\frac{1}{4\pi}\int_{X^{10}}\phi F_2\wedge I_8(R) \tag{4.18}$$

provided that

$$\frac{1}{4\pi}\int_{X^{10}}F_2\wedge I_8(R)\in\mathbb{Z} \tag{4.19}$$

since ϕ is 2π-periodic. Here F_2 is the curvature of the upper half plane, given in terms of the modular parameter by

$$F_2=\frac{id\bar{\tau}\wedge d\tau}{4(\operatorname{Im}\tau)^2}\,, \tag{4.20}$$

and $I_8(R)$ is the Green-Schwarz anomaly polynomial in R, the curvature of TX^{10}.

The SL$(2,\mathbb{Z})$ anomaly is cancelled by adding the term [21]

$$S''=\frac{i}{4\pi}\int_{X^{10}}\ln g(\tau)F_2\wedge I_8(R) \tag{4.21}$$

where $g(\tau)$ is a modular form that satisfies (up to a constant phase)

$$g(\Lambda\tau)=\left(\frac{c\tau+d}{c\bar{\tau}+d}\right)^{1/2}g(\tau) \tag{4.22}$$

where Λ is the SL$(2,\mathbb{Z})$ Möbius action. The SL$(2,\mathbb{Z})$ symmetry is unbroken if [21]

$$\frac{1}{4\pi}\int_{X^{10}}F_2\wedge I_8(R)\in N\mathbb{Z} \tag{4.23}$$

where N is 4 or 12 depending on the transformation property of $g(\tau)$. Therefore, we see that if we take the latter case then the integral

$$\frac{1}{2\pi}\int_{X^{10}}F_2\wedge I_8(R) \tag{4.24}$$

is in $24\mathbb{Z}$. If $I_8(R)$ is integral, then the U(1) curvature F_2 is in $24\mathbb{Z}$. We again see the mod 24 congruence.

4.4 The M-theory topological lagrangian

In [16], Witten derives the effective lagrangian of M-theory which comes from the Chern-Simons term. Simply to get consistency, i.e. to make the lagrangian well-defined, one gets the action (in our notation) (3.2) where Z^{12} is a *Spin*-manifold whose boundary is M-theory spacetime Y^{11}. In this section, we shall try to understand this lagrangian in the context of the kind of theories we are considering in this paper.

One has, (at least as differential forms),

$$d*G_4=-\frac{1}{2}G_4\wedge G_4\,. \tag{4.25}$$

It is therefore appealing to write the Chern-Simons lagrangian term ("on-shell") as

$$\frac{1}{12} G_4 \wedge (*G_4)\,, \tag{4.26}$$

which looks rather like a gauge-theoretical kinetic term. However note that this still does not explain the consistency of such expression.

But this is related to the mathematical notion of Massey products. A differential graded algebra (DGA) is a (not a priori commutative) graded algebra A with a map $d : A \to A$ of degree $+1$ which satisfies the relations

$$dd = 0\,, \tag{4.27}$$

$$d(ab) = (da)b + (-1)^{\mathrm{dim} a} a(db)\,. \tag{4.28}$$

(Different sign conventions are possible.) Then the cohomology $H(A)$ of A with respect to d is a graded algebra. It has further certain operations called (matrix) Massey products. These are essentially the only operations, but if A has any kind of commutativity property, more operations arise, although many of them are torsion. In any case, the simplest Massey product is a correspondence

$$H(A) \otimes H(A) \otimes H(A) \to H(A) \tag{4.29}$$

which is denoted by $[a, b, c]$, where $a, b, c \in H(A)$. It is defined only when $ab = bc = 0 \in H(A)$, and the dimension of the result is

$$\dim(a) + \dim(b) + \dim(c) - 1\,. \tag{4.30}$$

It is also not well defined, it is only defined modulo terms of the form $ax + yb$ where x, y are some elements of $H(A)$ over which we have no control. They may, however, sometimes be excluded, for example for reasons of dimension.

The definition of $[a, b, c]$ is as follows: we have

$$ab = dy\,, bc = dz \qquad \text{for } y, z \in A\,. \tag{4.31}$$

Then set

$$[a, b, c] = yc + (-1)^{\mathrm{dim} a+1} az\,. \tag{4.32}$$

It is obvious that this is a cocycle, and that the cohomology class is defined modulo the indeterminacy given above. It is worth noting that all Massey products are essentially elaborations of this principle. A Massey product $[a_1, \ldots, a_n]$ exists if and only if all "lower" Massey products of these elements vanish, and also one may do the same thing for matrices of elements. That is the whole story for DGA's.

In our situation, the equation (4.25) implies

$$-\frac{1}{2}[G_4, G_4, G_4] = [G_4, *G_4] \tag{4.33}$$

(the right hand side has the Lie bracket, the left hand side the Massey product). This suggests rewriting (3.2) as

$$\frac{1}{6}[G_4, G_4, G_4]\,, \tag{4.34}$$

which now is at least an expression which lives entirely in cohomology. However, let us take this one step further and see what are the implications of this in F-theory.

In [16], as noted above, Y^{11} is the boundary of a manifold, a 'Spin cobordism', Z^{12}. To prove invariance, one also considers the case when Z^{12} is obtained from gluing two cobordisms together, i.e. Z^{12} is an orientable compact manifold and Y^{11} is a submanifold of codimension 1 such that $Z - Y$ has two connected components (each of which is a cobordism). Then from the Mayer-Vietoris sequence, there is a connecting map

$$T : H^k(Y^{11}) \to H^{k+1}(Z^{12}) \tag{4.35}$$

(which can be thought of as a kind of transfer). Now let $a, b, c \in H^*(Z^{12})$ (we should think $a = b = c = G_4$). Suppose further $a'b' = b'c' = 0 \in H^*(Y)$ (the $'$ means restriction from H^*Z^{12} to H^*Y^{11}). Then we have

$$T[a', b', c'] = abc \quad \text{mod indeterminacy} \tag{4.36}$$

where the Massey product is taken in H^*Y, the product in H^*Z. The indeterminacy can be taken as $az+xc$ where z, x are cocycles in the opposite connected components of $Z^{12}-Y^{11}$.

A sketch of a proof can be obtained as follows: let us think of the Poincaré dual chains. Make the cycles representing a, b, c in Z^{12} intersect transversally with Y^{11}. Now restrict the chains a, b, c to chains (not cycles) a_i, b_i, c_i on the closures Z_i of connected components of $Z - Y, i = 1, 2$. Then $d(a_1b_1) = a'b'$, $d(b_2c_2) = b'c'$. Furthermore, in Z_1, the intersection of a_1b_1 with c_1 is the same as the restriction in Z_1 of u with c_1 where $du = a'b'$ in Y^{11}, which in turn is the same as the intersection of u with c' in Y. Similarly on Z_2. Now on chains, T is represented by inclusion. So

$$a_1b_1c_1 + a_2b_2c_2 \tag{4.37}$$

represents $T[a', b', c']$, as claimed.

This suggests again that the effective 12-dimensional topological lagrangian term should be

$$\int_{Z^{12}} \frac{1}{6} G_4 \wedge G_4 \wedge G_4 \tag{4.38}$$

with the indeterminacy described above, which of course coincides with the result of [16]. But more interestingly, the 1-loop correction term in (3.2) can be explained as a part of the indeterminacy of (4.36). Thus, it is interesting to note that indeed (4.34) is a correct way to rewrite (3.2), and that the 1-loop correction terms in M-theory is a part of Massey product indeterminacy.

Another comment is perhaps in order. It could be argued that a defect of the Massey product approach is that it does not specifically predict the 1-loop gravitational term as the correction term. This is a delicate issue and we would say this criticism is partially true: on the one hand, certainly the Massey product approach does not, without further rigidification of the input, predict the precise form of the counterterm. On the other hand, it does predict that such a term must exist.[7] (A caveat is the coefficient 1/6, which cannot be predicted by rational cohomology; a proper integral refinement, possibly using

[7] While there is a version of the Massey product which does not use indeterminacy, it requires more input data, and at present we do not know if it helps predict the one-loop term more accurately.

generalized cohomology, would be needed. Recall that the arguments applied in [4] are rather delicate. Although generalized cohomology is the main theme of this paper, and this is perhaps one of the fundamental issues of M-theory, we do not have this precisely worked yet.)

Accepting, however, that the Massey product does predict the existence of a counterterm, it is then actually not bad at predicting the term itself. The indeterminacy is

$$G_4 \wedge I_7 \tag{4.39}$$

where I_7 is a 7-dimensional cohomology class in Y^{11} (which must be distinguished from the 8-dimensional cohomology class I_8 in Z^{12}).[8] The term (4.39) does not appear to be excluded by the 1-loop approach. Although we do not know its exact meaning, it is probably related to the dynamics of M5-branes, as is the 1-loop term. In fact, (4.39) looks like a coupling of M2-brane and closed M5-brane field strengths.

One reason for discussing these manipulations here is that it is possible a similar device could be used to unify the two seemingly different F-theoy lagrangians (3.2), (3.3) in section 3. If we denote by G_5 the field strength corresponding to the potential A_4, the suggested F-theory topological term is

$$\frac{1}{6}\int_{V^{12}} [G_4, G_4, G_5]\,. \tag{4.40}$$

As written, the Massey product takes place in the algebra of differential forms.[9] This of course needs further discussion, but the point is both a 1-loop gravitational correction term and a term of the form (3.2) can be considered indeterminacy terms to (4.40). In the case of the 1-loop term, the discussion is similar as the case of M-theory earlier in this section. In the case of (3.2), this phase vanishes on a closed manifold $Z^{12} = V^{12}$. This indeed corresponds to adding the cocycle G_4 to the potential A_4, which is a gauge transformation not affecting the field strength.

5. The partition function of F-theory compactified on an elliptic curve

5.1 Elliptic cohomology

Let us now approach the problem from another angle. Namely, let us go back to 10-dimensional type II string theory. In [1, 2], we have observed that the partition functions of IIA and IIB string theories (see [4, 3]) can be lifted to elliptic cohomology. We constructed

[8] One might be tempted to say that $dI_7 = I_8$ via Stokes theorem. However, this is not correct for two reasons. First, I_8 is not a coboundary (although of course it is locally), and second, I_7 is *closed*, i.e. it is a cohomology class in $H^7(Y^{11})$. The situation is quite analogous to that of the potential/field strength: I_8 plays the role of the field strength, I_7 is the indeterminacy of the potential, which is a closed gauge term, i.e. a shift gauge transformation that can be added to it.

[9] Of course, a refinement of G_5 in elliptic cohomology of the 12-dimensional spacetime would be desirable. Schematically, this seems consistent since the dimension of the class would increase by 1 by wrapping around the additional degree of freedom. However, one must be careful while considering the exact nature of this additional dimension. We will return to this point later.

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this lifting by carefully observing the homotopical content of the IIA partition function obstruction of [4]. However, what is the correct interpretation of these partition functions?

In this section, we propose an answer to this question: the elliptic partition function belongs to F-theory compactified on an elliptic curve, which unifies both IIA and IIB string theories. Roughly, the idea is this: in elliptic cohomology, we see another parameter in the coefficients of the theory. In [1], we worked mostly with the cohomology theory $E(2)$, in which case the extra parameter will be $(v_1)^3(v_2)^{-1}$, where v_1 is the Bott generator and v_2 is the degree six analog. One can work with other elliptic spectra and get different parameters. But the point is that in all cases, the additional parameter is some modular form of some level, i.e. a power series in $q = e^{2\pi i\tau}$ where τ is the modular parameter of an elliptic curve. So one can ask what causes a theta function (more precisely theta constant) of a lattice Γ to be modified in this fashion, i.e. where the value is, instead of a number, a function in a modular parameter τ of an additional elliptic curve? The answer is that this arises precisely when we tensor Γ by another lattice of dimension 2 whose period is τ. Tensoring with a two dimensional lattice amounts to summing two copies of Γ (at least as abelian groups). One can argue that if $q \to 0$, then $\tau \to i\infty$, so the other copy of Γ is "infinitely far", thus reducing the new function to the old one in the $q \to 0$ limit.

But where does the new lattice come from? It comes from the 1st cohomology of an elliptic curve, which is the theory E on which we are compactifying F-theory. In other words, in F-theory on $V^{12} = X^{10} \times E$ which contains type IIB string theory on X^{10}, the odd degree field strengths move to even-dimensional cohomology of V^{12}, as predicted above in section 4.1. For example, from the F-theory term

$$\int_{V^{12}} A_4 \wedge G_4 \wedge G_4 \tag{5.1}$$

which was proposed in [11] and used in [12], we obtain the type IIB Chern-Simons term

$$\int_{X^{10}} A_4 \wedge F_3 \wedge H_3 \tag{5.2}$$

after reducing on the elliptic curve one step at a time to get H_3 and F_3 as results of contraction of one index of G_4, and A_4 remains the same. So G_5 is lifted by $H^0(E)$ and G_3 by $H^1(E)$. What about G_1? Note that the reason to consider F-theory in the first place was to try to interpret G_1 (i.e. the axion-dilaton combination) as the (non-constant) moduli of the elliptic curve. Thus we propose that G_1 is not lifted to F-theory but only shows up in ten dimensions upon compactifying on a nontrivial torus. This is compatible with [12] who consider a field content in twelve dimensions consisting of a metric, a dilaton, a four-form and a five-form field strengths, but no p-form field strengths ($p = 2, 3$) which would come from lifting G_1 via H^1 and H^2. Of course, there could be a nontrivial mixing between the dilaton in twelve dimensions and the dilaton coming from the moduli of the torus. This might not be surprising from a Kaluza-Klein point of view, but we do not explore it further as it would be outside the scope of the paper.

In IIA, this might seem more confusing, since we have a field G_4 in dimension 4 and H_3 in dimension 3. However, we think the answer has to be as follows. Once again, we should have compactified F-theory on $V^{12} = X^{10} \times E$. But this time, consider an intermediate step, M-theory compactified on $X^{10} \times S^1$. In this compactified M-theory, the H_3 picks up a dimension by multiplying with a first cohomology class of S^1 and is absorbed into G_4. In the F-theory considered here (the standard F-theory), indeed G_4 expands into G_5 by coupling with $H^1(E)$. On the other hand, the IIA-theoretical H_3 becomes absorbed in this G_5 also by coupling with $H^2(E)$. Thus, we see the same modularity (see below for more notes on modularity) as in the standard F-theory related to IIB, and since that theory has both G_4 and G_5, this further supports the idea that this theory be a unification between type IIA and IIB string theories (see [8, 11, 12]). However, we note from section 3 above that this sector is not obtained as cobordism of Y^{11}, but as $Y^{11} \times S^1$.

5.2 E-theoretic formula for the fields and new characteristic classes

According to [4], formula (7.2) states that the total field strength $G(x)$ of type II 10-dimensional string theory is 2π times

$$\sqrt{\hat{A}(X)}ch(x)\,. \tag{5.3}$$

This formula is needed, since the metric of the K-theory lattice is, up to a factor of $1/(2\pi)^2$, given by

$$\int_X G(x) \wedge *G(y)\,. \tag{5.4}$$

However, formula (5.3) applies to a K-theory setting, so it needs adjustment in case of elliptic cohomology. There is no problem with the Chern character, since for any elliptic cohomology theory E, there is a canonical map $E \to K((q))$ (where q is as above), so we may compose with the Chern character to get a map

$$ch_E : E \to H^*((q))\,. \tag{5.5}$$

On the other hand, the term $\sqrt{\hat{A}(X)}$ should be replaced by an analogous term related to the Witten genus, which is

$$\sigma(X)^{1/2} \tag{5.6}$$

where $\sigma(X)$ is the characteristic class of X associated with the power series

$$\sigma(z) = (e^{z/2} - e^{-z/2}) \prod_{n\geq 1} \frac{(1-q^n e^z)(1-q^n e^{-z})}{(1-q^n)^2}\,. \tag{5.7}$$

Therefore, our formula for the elliptic field strength associated with x is[10]

$$G(x) = \sigma(X)^{1/2} ch_E(X)\,. \tag{5.8}$$

Note that this σ-function, in the $q \to 0$ limit, reduces to the characteristic function of the $\hat{A}$-genus, thus reducing this field strength to the type II field strength in the 10-dimensional limit.

[10] We have demonstrated this formula only up to terms that vanish as $q \to 0$, so in principle such terms could be present. However, it is not obvious that there are natural such candidates.

We should of course remark that the partition function we consider is, similarly as in [4], approximate in that we work in the free field limit. This means that the action we consider is essentially just the hermitean metric on the field strengths. Using the standard definition of partition function, we therefore obtain the theta function.

The definition of the elliptic partition functions given in [1, 2] are then complete. As mentioned above, we propose that these functions are, in fact, partition functions of the F-theory sectors on $X^{10} \times E$ which, when E goes to 0, reduce to type IIA and IIB 10-dimensional string theories.

5.3 Interpretation of Twist and modularity

Let us recall now again the IIB string theory modularity puzzle (see section 4.1). In type IIB string theory, we have an RR-field strength F_3 and an NSNS-field strength H_3 which are in a relation of modularity. As pointed out in [4], the K-theory based partition function for type IIB does not explain that modularity, and it cannot be explained by twisted K-theory either, as shown in [2]. Of course, as also mentioned there, the possibility is not excluded that by introducing more terms, such as the P-term that depends only on the topology and the spin structure of the manifold [4, 38], into the modularity equation, one could start building by hand a Postnikov tower of a different classifying space or generalized cohomology which could give the correct explanation.

This is however not the approach we take here. Instead, we build directly a theory (at least its free field approximation) based on elliptic cohomology of the 10-dimensional spacetime X^{10}. What we conjecture (see also [2]) is that this theory is related to F-theory compactified on an elliptic curve E. It is rather natural then to conjecture that modularity in the first cohomology of E explains the modularity in type IIB theory. (We also commented briefly above on why this modularity is broken in IIA.) This construction does not come for free. In order for the lift to F-theory to be consistent, we get an obstruction

$$w_4 = 0 \tag{5.9}$$

which seems foreign to type II string theory (although it occurs in heterotic string theory, thus perhaps hinting that F-theory provides an even further unification). Also, the combined field strength G_3 (see section 4.1 above for more discussion) must be lifted to elliptic cohomology, which restricts the kind of configurations allowed. Twisting in the new theory disappears. The combined G_3 field strength is (an additive) generalized cohomology class, whereas twisting allows, at least a priori, non-additive configurations.

After introducing all this, we got modularity which is indeed tied to the modularity in the first cohomology of the F-theoretical fiber E. However, note that even then the picture we get is not quite as ideal as one might hope. Mathematically, the problem is that elliptic cohomology spectra are not completely modular with respect to the whole group $\mathrm{SL}(2,\mathbb{Z})$. Only the spectrum TMF enjoys such full modularity, but that is not an elliptic spectrum. In fact, if we agree to specialize to information at $p = 2$ (2-torsion does seem like the most interesting information), then following Hopkins and Mahowald [41], we may use the elliptic spectrum E_2 with coefficients $W_2[[a]][u, u^{-1}]$. Then TMF (at $p = 2$) can be obtained as homotopy fixed points of E_2 with respect to an action of the group $\mathrm{SL}(2, \mathbb{Z}/3)$. In other words, we may roughly say that E_2 is modular with respect to the congruence group $\Gamma(3)$, i.e. that we are only allowed to perform modular transformations which fix the group of points of order 3 on the elliptic curve. Accordingly, we only recover level 3 modularity of the combined G_3 field strength of type IIB string theory.

One could conjecture that an ideal F-theory (as was suggested above) could be obtained by an orbifolding analogous to the construction in homotopy theory which produces the spectrum TMF (or its connected form, tmf). Let us try to work out the implications of such construction. First of all, mathematically, we have the advantage that we have a toy model. To simplify the discussion, let us look again at generalized cohomology theories completed at $p = 2$, as in the last paragraph. Then we saw we get TMF from E_2 by taking homotopy fixed points with respect to the group $\mathrm{SL}(2, \mathbb{Z}/3)$. However, that group has a normal subgroup, namely the center, which is isomorphic to $\mathbb{Z}/2$. The non-zero element α of this center is the diagonal matrix with entries equal to -1. We can therefore obtain TMF in two stages, first taking homotopy fixed points

$$(E_2)^{h\mathbb{Z}/2}\,, \tag{5.10}$$

and then again homotopy fixed points of the generalized cohomology theory (=spectrum) (5.10) with respect to $\mathrm{PSL}(2, \mathbb{Z}/3)$. However, as noted above, the map α is the inverse operator on the elliptic curve (in homotopy theory, one sees a so called supersingular elliptic curve over $\mathbb{F}_4$, and in fact all its information is extracted from its formal group law, which is of height 2; see [1] for a review of formal group laws in the physical context. The element α is then the inverse series of that formal group law). The point of discussing this in such detail is that taking fixed points with respect to the inverse series of a formal group law is a well known operation in homotopy theory: one obtains the real form of the theory. For example, starting with K-theory, one obtains KO. Starting with E_2, the theory (5.10) becomes in fact the real elliptic cohomology theory

$$(E\mathbb{R}_2)^{\mathbb{Z}/2} \tag{5.11}$$

discussed in [45] (as shown there, there is a "completion theorem" which makes it unnecessary in this case to distinguish between actual and homotopy fixed points).

The appearance of the real form of a generalized cohomology theory is interesting here. In the case of K-theory, its real form KO describes the sources of type I string theory, which can be obtained from type IIB string theory by orbifolding. Note however that this is worldsheet orbifolding, using the automorphism of the theory which exchanges the chiral sector, i.e. a worldsheet involution that reverses the signs of the worldsheet coordinates, and thus interchanging left movers with right movers. It does not seem from this worldsheet point of view that in 10 dimensions, one could consistently orbifold any further. Another way of expressing this is to say that supersymmetry cannot be broken further than $N = 1$, starting from $N = 2$.

From the 12-dimensional point of view, when constructing (5.10), however, we see another side of the story. We can, in fact, identify physically what kind of orbifolding (5.10) corresponds to. This is because we know that the element α is the inverse of the elliptic curve, and that elliptic curve we understand (from modularity) to be a form of the fiber E of (4.5). Therefore, we are orbifolding with respect to the involution of the two fiber dimensions in spacetime! In this context, the additional orbifolding with respect to $\mathrm{PSL}(2, \mathbb{Z}/3)$ could possibly be consistent, although details would certainly have to be worked out. But how is it possible that worldsheet orbifolding of type IIB in 10 dimensions could correspond to the spacetime fiber orbifolding in dimension 12?

While we do not have a complete explanation (it is perhaps a “string miracle”), we can point out that this phenomenon, at least in 11 dimensions, has in some sense already been observed. Compactified M-theory on S^1 is on the strong/weak duality line between type IIA string theory and M-theory. Applying spacetime orbifolding to the eleventh dimension with respect to the inverse operator gives Hořava-Witten M-theory, which is S-dual to $E_8 \times E_8$ heterotic string theory. Applying T-duality, we get $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string theory, which is S-dual to type I string theory. The latter is obtained by worldsheet orbifolding of type IIB string theory [39] via projecting by an involution (i.e. orientifold) Ω that exchanges the left and the right closed string oscillators and acts on the open string oscillators by introducing a $\mathbb{Z}_2$ phase.[11] We propose that type I can be lifted to a theory $\widetilde{\mathrm{M}}$ which is T-dual to the original M-theory compactified on S^1 (as remarked below, we do not know if $\widetilde{\mathrm{M}} = M$). In any case, if we suppress U-dualities from the notation, we get, schematically, the following diagram:

$$\begin{tikzcd} * \arrow[r, "sO"] \arrow[d, "T"'] & * \arrow[d, "T"] \\ * \arrow[r, "wO"] & * \end{tikzcd}$$

where sO stands for spacetime orbifolding in the eleventh dimension, and wO stands for worldsheet orbifolding in 10 dimensions, and T stands for T-duality. This is the kind of relation between worldsheet and spacetime orbifolding proposed above.

Note that the spacetime involution applies to both dimensions of E, so it preserves orientation, while the worldsheet involution only applies to one coordinate, i.e. it reverses orientation.

One might also justify the truncations done in [12] for the reduction from F-theory to M-theory and type IIB string theory. There, (consistent) truncations were imposed by hand on the fields, which amounted to setting G_5 to zero in compactifying to M-theory on S^1 and setting G_4 to zero in compactifying F-theory to type IIB string theory on an elliptic curve. We propose that such truncations can be made natural by looking at a Hořava-Witten-like construction, but for the elliptic curves instead of the circle. More precisely, we propose the existence of involutions on both elliptic curves, the one fibered over IIA and the one fibered over IIB, in such a way that orientation-reversing kills G_4 in the case of IIB and kills G_5 in the case of IIA. From the point of view of M-theory, this

[11]More precisely, Ω acts on the closed sector by exchanging α^μ_m and $\tilde{\alpha}^\mu_m$ and on the open sector by exchanging α^μ_m and $\pm(-1)^m \alpha^\mu_m$.

means that the extra twelve-dimensional circle comes with an involution on it. [12] We do not attempt a construction here as this would be beyond the scope of this paper.

Let us make one more comment, which is more related to the IIA sector. Diaconescu, Freed and Moore [38] consider a cubic refinement of the triple pairing in G_4 associated with M-theory. This in our language is related to the cubic structure [40] on elliptic cohomology in the same way as the quadratic refinement of the pairing ω in [4] is related to the quadratic structure on K-theory corresponding to KO.

5.4 The Witten genus and a possible explanation via loop groups

There is another provocative coincidence which may support our explanation of the elliptic partition function as compactification of the standard F-theory (3.3) on an elliptic curve. When considering the elliptic field strength (5.8), we see that the free action (hermitean metric) in that theory relates to the Witten genus in the same way as the action of the K-theoretical field strength [4] relates to the $\hat{A}$-genus.

But when Witten first introduced his genus [10], he made another suggestion of relation with the $\hat{A}$-genus, namely that his genus should be related to taking index of loop bundles on loop space. This, in fact, has led to much speculation on the nature of elliptic cohomology, which is well summarized in [42] (see the volume [43] for the orginal references). Most of this speculation, which continued to the present day (cf. [44]), was in the worldsheet modularity direction, but when trying to match this with evidence from loop groups, [9] found that the elliptic curve shows up in spacetime as well. Here we shall propose that the elliptic curve in spacetime should, in fact, be the fiber of standard F-theory compactified on the elliptic curve. In fact, strikingly, [9] found defects to modularity very similar to those found in the present paper.

What we propose is the following. When forming the compactification of F-theory on an elliptic curve, there is an intermediate step: compactification of F-theory on a circle, which should be M-theory following [12]. In view of our previous discussion in section 3, it is safest here to consider this as a sector of F-theory which contains type IIA string theory; the sector containing IIB-theory may possibly be different. In fact, in the IIB case, one should also have an S^1-reduction of F-theory, and one can have a symmetric picture between type IIA and type IIB string theory in connection to F-theory. Another way to pose this question is whether it makes sense to ask for a “T-dual” of M-theory. We do not know if M-theory would be “T-dual” to itself, although this seems to be hinted at in [8]. We sometimes use the term $\widetilde{\text{M}}$-theory to refer to the “T-dual” of M-theory which contains IIB. We do not know if $\tilde{M}$-theory is the same as M-theory under suitable conditions. However, the existence of such a theory would not follow from a strong coupling argument, since IIB is S-selfdual, and thus the arguement could be similar to the one used for going from M-theory to F-theory.

[12]We do not imply that all the theories we discuss are related either by taking boundaries or simply by using S^1 factors. The discussion is schematic and a more precise description will involve other manifolds such as $K3$ in order for the picture to be compatible with the web of dualities. In some limit, $K3$ can be viewed as some orbifold of T^4 so what we mean by a circle with an involution is something crudely similar within the Calabi-Yau manifolds.

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Now looking at the IIA picture, M-theory on an 11-dimensional manifold Y^{11} vs. F-theory [12] compactified on $Y^{11} \times S^1$. If we look at the lagrangian term (3.3), then it suggests that we can in fact express the whole $Y^{11} \times S^1$-state by a state on Y^{11}, valued not in the original target space, but in the loop space of that space. Therefore, one could conjecture that the effective action of F-theory compactified on S^1 can be computed in the same index-theoretical way as the effective action of M-theory, but instead of the E_8 and Rarita-Schwinger index terms, one would substitute loop bundle versions of those indices. Now note that the index of such operators should also be taken on loop space, but it is well known that the relevant homotopical information is contained at constant loops (this is well explained in [42]), so integrating again over Y^{11} seems adequate.

But now following [42], replacing Dirac operators by the corresponding Dirac operators on loop bundle should correspond to replacing the $\hat{A}$-genus by the Witten genus in the answer. Thus, this suggests a modification of the method of [4] to compare genuine F-theory partition function obtained from its kinetic term, via its interpretation as loop-bundle index on the spacetime Y^{11} of M-theory, to the elliptic 'Witten genus' modification of the K-theoretical partition function described in in detail above in this section. We do not carry out this calculation here in detail, but propose it as a concrete calculational experiment which could be used to test whether the field theory [12] really has a consistent compactification to M-theory.

6. Remarks on signatures and supersymmetry

Note however first that all the homotopy theory work seriously described in this paper is done in euclidean signature. To discuss signatures seriously, we need to adapt our discussion to manifolds with signatures. Here we simply point out the relevance of signatures.

As far as generalized cohomology with signatures, not much has been done. Manifolds with signature typically cannot be compact, so we must take cohomology with compact supports. But how to take the signature into account in generalized cohomology? A suggestive point is that KO-theory $KO^{p,q} \cong KO^{p-q}$ looks like it should be KO-theory of spacetime with signature (p,q). This, indeed, suggests a proposal: ordinary cohomology, K-theory and elliptic cohomology are all $\mathbb{Z}/2$-equivariant generalized cohomology theories, which we can interpret as generalized cohomology theories with a real form ([45]). Now if M is a manifold with signature, this makes the tangent bundle TM a $\mathbb{Z}/2$-equivariant bundle, where $\mathbb{Z}/2$ reverses signs of purely time-like dimensions (this is not completely Lorentz-invariant, but is so up to homotopy). Let us call this new $\mathbb{Z}/2$-equivariant structure on the tangent bundle TM_ϵ. Then we can define, for a real-oriented generalized cohomology ER, the signature-cohomology of M as

$$ER_c^k(TM_\epsilon)\,, \qquad k \in \mathbb{Z} \tag{6.1}$$

where c denotes compact support. This is, of course, still a long way from working out all the homotopy theory we have above at signatures, but it is a start. We will develop the theory further elsewhere.

We start with by looking at Clifford algebras in twelve and eleven dimensions with various signatures. A discussion on spinors in different dimensions and with various signatures can be found in [46]. In twelve dimensions, we are interested in (s,t) signatures, with $t = 0,1,2,3$. The corresponding Clifford algebras are isomorphic to the matrix algebras

$$(12,0) : \mathrm{Mat}_{32}(\mathbb{H}) \tag{6.2}$$
$$(11,1) : \mathrm{Mat}_{32}(\mathbb{H}) \tag{6.3}$$
$$(10,2) : \mathrm{Mat}_{64}(\mathbb{R}) \tag{6.4}$$
$$(9,3) : \mathrm{Mat}_{64}(\mathbb{R}) \tag{6.5}$$

so that the spinor representations are quaternionic in the first two cases and real in the last two cases. For the spinor representation, one has to look at the even Clifford algebra which is given by

$$Cl(s,t)^{even} \cong Cl(s-1,t) \qquad \text{for } s \geq 1\,. \tag{6.6}$$

Then the even Clifford algebras are given by

$$(12,0) : \mathrm{Mat}_{16}(\mathbb{H}) \oplus \mathrm{Mat}_{16}(\mathbb{H}) \tag{6.7}$$
$$(11,1) : \mathrm{Mat}_{32}(\mathbb{C}) \tag{6.8}$$
$$(10,2) : \mathrm{Mat}_{32}(\mathbb{R}) \oplus \mathrm{Mat}_{32}(\mathbb{R}) \tag{6.9}$$
$$(9,3) : \mathrm{Mat}_{32}(\mathbb{C})\,. \tag{6.10}$$

So one can have the following types of spinors in twelve dimensions

$$(12,0) : \text{Symplectic Majorana-Weyl} \tag{6.11}$$
$$(11,1) : \text{Majorana} \tag{6.12}$$
$$(10,2) : \text{Majorana-Weyl} \tag{6.13}$$
$$(9,3) : \text{Symplectic Majorana}\,. \tag{6.14}$$

For the lorentzian case, $(11,1)$, we have Majorana spinors. In this case, one can try to form a supermultiplet for supergravity formed out of 320 bosons and 320 fermions, but the gravitino and the form sectors of the structure are incompatible [47]. One can then ask whether one can construct supergravity theories with other signatures in twelve dimensions. A general discussion on this can be found in [48], and a proposal in the $(10,2)$ signature can be found in [49, 50]. Note that for $(9,3)$ we can have symplectic-Majorana spinors, whose defining relations for the charge-conjugation matrix C and the gamma matrices γ^μ are given by

$$C^T = -C \qquad (\gamma^\mu C)^T = +\gamma^\mu C \qquad \gamma^{\mu T} = -C^{-1}\gamma^\mu C\,. \tag{6.15}$$

Some more discussion on this from point of view of physics as well as mathematics will be discussed seperately.

Let us however make one final remark on a possible significance of the signatures in connection with the IIA/IIB duality. In the $(10,2)$ signature, the fiber is a lorentzian torus, which seems to break modularity. On the other hand, this model seems forced if we want a physical version of the proposal of [12] (since signature $(9,3)$ does not contain $(10,1)$, which is the physical signature of M-theory). This could be consistent, since in type IIA, over which this sector of F-theory is fibered, we indeed do not have manifest modularity.

On the other hand, in type IIB theory, we need manifest modularity, so it seems that physically, the $(9,3)$-sector is required. However, now this sector of F-theory cannot contain a physical $(10,1)$-M-theory, which again seems consistent, as IIB theory does not seem to have a $(10,1)$-M-theory dimensional expansion. It is possible that a $(9,2)$ expansion is possible, and that this could in fact be the correct physical signature for $\widetilde{M}$-theory. This might be not so unreasonable since there are versions of eleven-dimensional M-theory in signatures $(1,10)$, $(2,9)$, $(5,6)$, $(6,5)$, $(9,2)$, and $(10,1)$ [51–53]. In fact in those theories, one already sees a difference between type IIA and type IIB theories: while IIA allows for both $(10,0)$ and $(9,1)$ signatures, type IIB allows for $(9,1)$ but not $(10,0)$.

A. A brief review of topological modular forms.

To make this paper more self-contained, we give here a very brief review of the theories tmf and TMF. This theory is due to Mike Hopkins and Haynes Miller. All information necessary for our purposes can be essentially found in [41]. The main point is this: in homotopy theory, it is convenient to consider multiplicative (commutative associative) generalized cohomology theories E (also called spectra) which are 2-periodic (in the same way as K-theory), and are complex oriented, which means that the generalized cohomology of the complex projective space is of the form

$$E^*(\mathbb{C}P^\infty) = E^*[[x]] \tag{A.1}$$

where x is the E^*-valued 1-st Chern class of the universal line bundle (equivalently, it suffices to say that such Chern class exists). It then follows that all complex bundles have E^*-valued Chern classes. In particular, one has

$$E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}P^\infty) = E^*[[1\otimes x, x\otimes 1]] = E^*[[y,z]]\,. \tag{A.2}$$

The multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ (classifying tensor product of line bundles) then gives, via (A.1), (A.2), a map

$$E^*[[x]] \to E^*[[y,z]]\,,$$

and the image of x under this map is a series $F(y,z)$ called a (1-dimensional commutative) formal group law (abbr. FGL). Its properties are

$$\begin{aligned} F(x,0) &= x\,,\\ F(x,y) &= F(y,x)\,,\\ F(x,F(y,z)) &= F(F(x,y),z)\,. \end{aligned}$$

Note that this looks like the properties of an analytic parametric expansion of the multiplication in a 1-dimensional commutative Lie group. That is not very interesting, of course, since all such groups are additive. Accordingly, even more generally, over a field of characteristic 0, all FGL's are isomorphic. However, the essential point is that FGL's can be considered over any commutative ring, and then this isomorphism statement is no longer true. In fact, much information about a complex oriented generalized cohomology theory can be deduced from its FGL. In particular, the Lie group construction can be extended to 1-dimensional commutative algebraic groups, and this includes, in addition to the additive and multiplicative group, also elliptic curves. In fact, in the case of elliptic curves, it can be extended even further, to generalized elliptic curves, which only have multiplication defined in a Zariski neighborhood of the identity. Details are irrelevant here (more precisely, are for our purposes subsumed by what we shall say next). A complex-oriented 2-periodic spectrum whose FGL is isomorphic to that of a generalized elliptic curve by a given isomorphism is called an *elliptic spectrum.* (To be completely precise, it is in fact convenient to add another condition that all coefficient groups of elliptic spectra are in even dimensions.)

Now algebraic geometers had long had to cope with the fact that there is not, in the proper sense, a universal generalized elliptic curve (for the same reason, there is also not a universal elliptic cohomology theory), although the problem only arises at the primes 2, 3. What there is, however, is the Weierstrass curve, which is written, in affine coordinates, as

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{A.3}$$

The coordinate transformations allowed are

$$\begin{aligned} x &= x' + r\,, \\ y &= y' + sx' + t\,. \end{aligned}$$

Transformations for the a_i's are easily deduced, but we do not need to write them down for our purposes. The outcome is that we obtain the pair

$$(A, \Lambda) = (\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \mathbb{Z}[a_1, \ldots a_6, r, s, t])\ . \tag{A.4}$$

Here one should think of r, s, t as free variables, i.e. polynomial generators. When representing an actual reparametrization of a generalized elliptic curve, they would take values in the ring of definition of the curve. The main point is that although every generalized elliptic curve is essentially a Weierstrass curve (i.e. can be obtained by choosing the a_1's appropriately in an appropriate commutative ring), the pair (A.4) does not have the structure of coefficient rings of a group scheme, thereby confirming that there indeed cannot be a universal (generalized) elliptic curve. However, (A.4) satisfy the axioms of what is called an affine algebraic groupoid (or, in homotopy theory, often Hopf algebroid). This proves that there is a *Deligne-Mumford stack* of generalized elliptic curves.

Tensoring (A.4) with $\mathbb{Z}[u, u^{-1}]$ where u is an element of dimension 2, we get $(A[u, u^{-1}], \Lambda[u, u^{-1}])$. These graded rings can be realized as coefficient rings of generalized elliptic spectra. Now all of the difficulty of the construction of tmf is contained in the statement that the structure maps of (A.4) (i.e. the maps realizing its structure as an affine algebraic groupoid) can be realized by maps of spectra (in particular generalized cohomology theories). In fact, more is true, it can be generalized by maps of E_∞-ring spectra, which are commutative associative ring cohomology theories in a particularly strong sense.

The spectrum tmf is then defined as the homotopy inverse limit of these structure maps, or equivalently of the system of all E_∞ elliptic spectra with respect to E_∞ maps coming from morphisms of generalized elliptic curves. This construction was carried out in detail by Hopkins and Miller, and recently much simplified by Jacob Lurie, using a remarkable approach to algebraic geometry directly in the category of E_∞ ring spectra.

Now just as there is no universal generalized elliptic curve, there is no universal elliptic spectrum, so accordingly, tmf is not an elliptic spectrum. However, its coefficient groups map to modular forms, and are called *topological modular forms*. Not every form is a topological modular form, and there are also topological modular forms which are 0 as ordinary modular forms. In particular, the discriminant form Δ is not a topological modular form, but its 24'th power is. It is some times convenient to invert this 24'th power, thereby obtaining a 576-periodic spectrum, which is denoted by TMF.

As we mentioned above, all the subtlety of TMF is at the primes $2, 3$. When inverting $2, 3$ in TMF_*, we obtain simply ordinary modular forms:

$$\mathbb{Z}[1/6][g_2, g_3][\Delta^{-1}]\,.$$

Completing at the prime 2 (which is a slightly stronger operation than localizing), the calculation of the homotopy groups of tmf is carried out in [41]. There, one can in fact say that there is a universal curve (with automorphisms). Its formal group law is the Lubin-Tate law of height 2. The curve can be taken to be the curve $x^3 + y^2 + y = 0$ over the 4-element field $\mathbb{F}_4$, and its group of rational points is $\mathbb{Z}/3 \times \mathbb{Z}/3$. We see there is a remarkable coincidence here with modular forms of height 3 over $\mathbb{C}$, which in fact plays a major role in mathematics, but we do not need to consider this in detail for the purposes of the present paper.

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